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$O(4)$ Treatment of Arbitrary Central Problem via Quasiclassical Quantization

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Abstract

A complete algebraic treatment of quantum problems with arbitrary central potential is given in terms of broken $O(4)$ invariance. This is done within the quasiclassical accuracy. More precisely, a generalised Runge–Lenz (RL) vector is built as a non-conserving member of the $O(4)$ Poisson brackets algebra. Its algebraic partner is the orbital momentum. The requirements imposed on the RL vector are as follows: (i) its length is conserved, (ii) it rotates following the procession of the orbit, (iii) it vanishes for the circular motion. These requirements suffice for unique determination of the RL vector. This enables us to express the Hamiltonian of an arbitrary central problem as a function of the $O(4)$ Casimir invariant and the angular momentum squared. The dependence upon the latter (generally very complicated) describes the way in which the symmetry is broken for a given potential. Replacing the $O(4)$ Casimir operator and the angular momentum by their known eigenvalues results in the Bohr–Sommerfeld quantisation rules. $O(4)$ multiplets of energy levels free of angular momentum degeneracy are described, examples are considered, and the inverse problem is discussed.

1. *Introduction*

We suggest a scheme based on the Lie algebra of the $O(4)$ group that provides a framework for handling three dimensional quantum problems with arbitrary central attractive potential. The common feature of the present approach and of those in the literature (Barut, 1964; Budini, 1966; Mukunda *et al.*, 1965) is the use of non-conserving generators, which are necessary if one deals with the problems free from the corresponding symmetry group. We describe the bounded motion in an arbitrary central field of force in terms of the broken $O(4)$ symmetry, the way it is broken being determined by the properties of the potential. The hydrogen atom exhibits the exceptional case when the breaking disappears in accordance with the well-known result of Fock (1935) and Bargmann (1936). Our scheme, when realised, implies the possibility of re-expressing the Hamiltonian, $H = (p^2/2) + V(r)$, in terms of the integrals of motion: the Casimir invariant and the angular momentum squared. The discrete energy

spectrum classification may be given according to irreducible representations of the $O(4)$ group.

In Section 2 we present our broken $O(4)$ symmetry scheme using the language of the Poisson brackets relation between classical functions, since this is the only thing we are able in fact to realise. We further formulate the problem of quasiclassical quantisation that is reduced to replacement of the Casimir invariant and angular momentum squared by their known eigenvalues in the classical expression of the Hamiltonian. In Section 3 we build the $O(4)$ generators obeying the scheme of Section 2 as explicit classical functions of position and momentum of the particle moving in a given central field of force. The problem of making these generators single-valued is solved in Section 4 by considering their connection with the geometry of the nonstationary orbit of the particle. As a result we find a closed expression for the Casimir invariant in terms of the energy and the orbital momentum squared. The quantisation prescriptions of Section 2 now lead to the Bohr–Sommerfeld rules of the old quantum theory. In Section 5 several potentials are considered as examples (the square well potential being among them). In Section 6 the inverse problem is outlined: how the potential can be restored if the Hamiltonian is given as a function of the Casimir invariant and the orbital momentum squared.

2. *Formulation of the Approach*

Let us have a Hamiltonian

$$H = \frac{p^2}{2} + V(r) \tag{2.1}$$

as a classical, spherically symmetric, function of momentum $\mathbf{p}$ and position $\mathbf{r}$. Let, together with the conserving angular momentum vector $\mathbf{L} = \mathbf{r} \times \mathbf{p}$,

$$\{L_i, H\} = 0, \qquad i = 1, 2, 3 \tag{2.2}$$

a (generally) non-conserving vector $\mathbf{A}$ exist:

$$\{A_i, H\} \neq 0, \qquad i = 1, 2, 3 \tag{2.3}$$

We call it the Runge–Lenz vector in analogy with the hydrogen atom. It will also be expressed as a function of $\mathbf{r}$ and $\mathbf{p}$. Let the components of $\mathbf{A}$ and $\mathbf{L}$ form the $O(4)$ algebra

$$\{A_i, A_j\} = \varepsilon_{ijk} L_k, \{L_i, A_j\} = \varepsilon_{ijk} A_k, \{L_i, L_j\} = \varepsilon_{ijk} L_k \tag{2.4}$$

The curly brackets in equations (2.2), (2.3), (2.4) and in what follows are the Poisson brackets, ε_{ijk} is the completely antisymmetric unit tensor. The quantity $L^2 = L_i^2$ is the Casimir invariant of the $O(3)$ subalgebra: $\{L^2, L_i\} = 0$. One can also form two Casimir invariants $G_{1,2}$ of the $O(4)$ algebra according to the known rules:

$$G_1 = \tfrac{1}{2}(L^2 + A^2) \tag{2.5}$$

$$G_2 = (\mathbf{LA}) \tag{2.6}$$

that commute with all the $O(4)$ generators:

$$\{G_{1,2}, A_i\} = 0, \qquad \{G_{1,2}, L_i\} = 0 \tag{2.7}$$

We confine ourselves to considering the vector **A** lying in the plane of motion, i.e. in the $(\mathbf{r},\mathbf{p})$-plane. Hence

$$G_2 = 0 \tag{2.8}$$

and we shall usually speak about a single Casimir invariant (2.5). (The subscript '1' will be omitted everywhere below.)

Demand that the Casimir invariant be an integral of motion

$$\{G, H\} = 0 \tag{2.9}$$

According to equations (2.5), (2.2) this means the constancy of the length of the Runge–Lenz vector $\{A^2, H\} = 0$. The quantities G, L^2 and, e.g. L_3, may form a 'complete set of commuting integrals of motion', and the Hamiltonian is nothing but a function of them, $H = H(G, L^2, L_3)$ (see Appendix for the proof, if necessary). Now the spherical symmetry (2.2), together with the relations $\{G, L_i\} = \{L^2, L_i\} = 0$, forbids the L_3-dependence of the Hamiltonian and we are left with

$$H = H(G, L^2) \tag{2.10}$$

In the case of non-broken $O(4)$ symmetry one has $\{A_i, H\} = 0$, instead of equation (2.3) and H does not depend upon L^2, since $\{G, A_i\} = 0, \{L^2, A_i\} \neq 0$. This is the 'accidental' degeneracy of the hydrogen atom. In Section 4 we shall find the function (2.10) as a functional of the potential. The L^2-dependence of it will determine how specifically the $O(4)$ symmetry is broken for one or another potential.

We emphasise that the set of conditions (2.2)–(2.9) implies no restrictions on the Hamiltonian (2.1). Just the opposite is true: after we built an explicit expression for **A** in Section 3 we saw that it contains too much arbitrariness. The arbitrariness is so strong that the function (2.10) remains completely undetermined. This means that the algebraic scheme described above does not, by itself, contain sufficient dynamical information, this fact not being unexpected. In Section 4 we overcome this ambiguity by imposing two additional requirements on the Runge–Lenz vector, which in some sense relate this vector to the moving particle. This will enable us to represent the Hamiltonian (2.1) uniquely in the form (2.10).

Keeping this in mind we shall now formulate the problem of quantisation consistent with the algebraic scheme described.

Consider an operator realisation of the Poisson brackets $O(4)$ algebra (2.4), (2.7) and adopt (2.10) as an operator function for H. Then G, L^2, L_3 and H have a common set $\psi_{n,l,m}$ of eigenvectors, labelled by the eigenvalues associated with the Casimir operators of $O(4)$ (G) and its subgroups (L^2, L_3):

$$G\psi_{n,l,m} = h^2 \frac{n^2 - 1}{2} \psi_{n,l,m}; \qquad n = 1, 2, 3 \ldots,$$

$$L^2\psi_{n,l,m} = h^2 l(l+1)\psi_{n,l,m}; \qquad l = 0,1,2\ldots(n-1)$$
$$L_3\psi_{n,l,m} = hm\psi_{n,l,m}; \qquad -l \leqslant m \leqslant l,$$
$$H(G,L^2)\psi_{n,l,m} = H\left(\frac{h^2(n^2-1)}{2}, h^2 l(l+1)\right)\psi_{n,l,m} = E_{n,l}\psi_{n,l,m} \quad (2.11)$$

Here we have taken into account the fact that the commutativity between the Casimir operators and generators survives under multiplication by a constant factor h which should have, in (2.11), the dimensionality of Planck's constant and is taken as equal to the latter in accordance with the usual quantisation prescriptions.

One should be aware that, generally, the above operators are only approximately (with the accuracy of h) equal to those which may be obtained by naive replacement of $\mathbf{r}$ and $\mathbf{p}$ by the quantum operators $\hat{\mathbf{r}}$ and $\hat{\mathbf{p}}$, with the commutation relations $[\hat{r}_i, \hat{p}_j] = ih\delta_{ij}$ in the classical functions representing $\mathbf{L}$, $\mathbf{A}$, G, H. Therefore the energy spectrum to be obtained by substitution of $h^2(n^2-1)$ and $h^2 l(l+1)$ for $2G$ and L^2, respectively, in the above function $H(G,L^2)$, has a quasiclassical nature. With our determination of $H(G,L^2)$ (Section 4) the quantisation rules arising are essentially those of Bohr and Sommerfeld.

The energy levels classification is based on the fact that the set of $\sum_{l=0}^{n-1}(2l+1) = n^2$ vectors $\psi_{n,l,m}$ forms a basis of the irreducible representation of the $O(4)$ group $\mathcal{D}[(n-1)/2, (n-1)/2]$, i.e. the corresponding n^2 energy levels $E_{n,l}$ are joined into one $O(4)$ multiplet. Among these n^2 levels there is an ordinary magnetic quantum number degeneracy. There remains n $O(3)$ submultiplets with different energy: the orbital momentum degeneracy does not, generally, occur within the same $O(4)$-multiplet. The quantum number n is connected with the radial quantum number n_r involved in the Bohr–Sommerfeld quantisation and corresponding to the number of zeros of the Schrödinger wave function by the same formula as the hydrogen principle quantum number is:

$$n = n_r + l + 1 \quad (2.12)$$

All levels with different n_r and l, but with the same n, are joined into one $O(4)$ multiplet. This classification is valid for every central potential. An important case of harmonic oscillator is especially traced in Section 5. For problems with finite number of bound states the multiplets may be occupied either by stable or by quasistable states. (Note the difference with Mukunda *et al.* (1965).)

3. *Non-conserving Runge–Lenz Vector*

The vector obeying the conditions (2.4), (2.7), (2.8), (2.9) can be looked for in the following form

$$\mathbf{A} = R(r,L^2,H)\frac{\mathbf{r}}{r} + W(r,L^2,H)\frac{\mathbf{r}\times\mathbf{L}}{rL} \quad (3.1)$$

where R and W are unknown scalar functions of the position vector length r, angular momentum squared L^2 and the energy $H=(p^2/2)+V(r)$. (This is equivalent to dependence upon the three independent scalars r, p, $\mathbf{rp}$.) The unit vectors in (3.1) are mutually orthogonal and $L\equiv\sqrt{L^2}$. Equation (3.1) is the most general expression for the vector located in the plane of motion, the fulfilment of equation (2.8) being thus guaranteed. It was explained in the previous section that the Casimir invariant is a function of the energy and angular momentum squared. Thus the first of equations (2.7) reads

$$\{\mathbf{A}, G(H,L^2)\}=0 \tag{3.2}$$

Since the brackets $\{G,L^2\}$ and $\{G,H\}$ vanish one has to differentiate R and W in (3.1) only with respect to r at fixed values of H and L^2 when calculating the bracket (3.2). Thus equation (3.2) takes the form

$$\frac{\mathbf{r}}{r}\left[\left(\frac{(\mathbf{rp})}{r}\frac{dR}{dr}+\frac{L}{r^2}W\right)\frac{\partial G}{\partial H}+2LW\frac{\partial G}{\partial L^2}\right]+ \\ +\frac{\mathbf{r}\times\mathbf{L}}{rL}\left[\left(-\frac{L}{r^2}R+\frac{(\mathbf{rp})}{r}\frac{dW}{dr}\right)\frac{\partial G}{\partial H}-2LR\frac{\partial G}{\partial L^2}\right]=0 \tag{3.3}$$

where $(\mathbf{rp})/r=\sqrt{[2(H-V(r))-(L^2/r^2)]}$. Equating the expressions in the square brackets with zero one reduces (3.3) to the set of two first-order differential equations for R and W with L^2 and H as constant parameters. This set can be solved and we finally have for (3.1)

$$\mathbf{A}=\left(\frac{\mathbf{r}}{r}\cos\xi+\frac{\mathbf{r}\times\mathbf{L}}{rL}\sin\xi\right)C(H,L^2) \tag{3.4}$$

where

$$\xi=\int_{r_1(H,L^2)}^{r}\frac{1-2Fr^2}{r^2}\left(\frac{2(H-V(r))}{L^2}-\frac{1}{r^2}\right)^{-1/2}dr \tag{3.5}$$

and F denotes the following ratio

$$F=-\frac{\partial G/\partial L^2}{\partial G/\partial H} \tag{3.6}$$

In equations (3.4), (3.5) $C(H,L^2)$ and $r_1(H,L^2)$ are two arbitrary functions. The angle (3.5) may be also represented as $\xi=\phi-2FLt$ in terms of the angular position ϕ of the particle and the corresponding time t, with the initial conditions: $t=0$, $\phi=0$, $r=r_1$.

The Runge–Lenz vector (3.4), (3.5), (3.6) satisfies equation (2.4) if G is represented as (2.5) with no further restrictions imposed on it. This is natural since the second of the relations (2.4) is a consequence of the third one and merely shows the way a vector transforms under the infinitesimal space rotation produced by $\mathbf{L}$. The first of the relations (2.4) can be easily shown to follow from the rest of them and from equation (2.7), (2.8) if one

takes into account that all the vector and tensor quantities must be constructed from $\mathbf{r}$, $\mathbf{p}$, δ_{ij} and ε_{ijk}.

Equation (2.3) for the vector (3.4) looks like

$$\{A_i, H\} = 2FL\left(\frac{\mathbf{r}}{r}\sin\xi - \frac{\mathbf{r}\times\mathbf{L}}{rL}\cos\xi\right) \tag{3.7}$$

It can be seen from equations (3.7), (3.6) that the vanishing of F means the conservation of $\mathbf{A}$ and corresponds to the L^2-independent Casimir invariant (2.5) in agreement with that the fact mentioned in Section 2. The $F=0$ limit of equation (3.5) reduces expression (3.4) to Fradkin's (1967) integral of motion (see also Bacry *et al.* (1966) and Mukunda (1967). We studied this case in our previous paper (Serebrennikov & Shabad, 1971).

The length squared of the Runge–Lenz vector (3.4) is

$$A^2 = C^2(H, L^2) = 2G - L^2 \tag{3.8}$$

the Casimir invariant (2.5) being thus completely undetermined. This is the arbitrariness mentioned in Section 2. We shall, in Section 4, fix all the arbitrary quantities involved in equations (3.4), (3.5).

4. *Incorporating of Dynamics*

In order to attribute dynamical significance to the nonconserving Runge–Lenz generator, built in the previous section, it is necessary to connect it with the geometry of the particle motion. Restrict first of all the nonconservation (2.3) of the Runge–Lenz vector. Remember that the lack of the $O(4)$ symmetry in an arbitrary central problem is due to the unclosedness of the orbits of classical bounded motion. Thus the nonconservation of the Runge–Lenz vector (i.e. the breakdown of the $O(4)$ symmetry) is to be responsible for the precession of the orbit. This will be provided if one demands that

$$\int_{r_{\min}}^{r_{\max}} \frac{1-2Fr^2}{r^2}\left(\frac{2(H-V(r))}{L^2} - \frac{1}{r^2}\right)^{-1/2} dr = \pi \tag{4.1}$$

where the radial coordinates of the turn points $r_{\min,\max}(H, L^2)$ are the two solutions of the equation

$$2(H - V(r))r^2 - L^2 = 0 \tag{4.2}$$

Requirement (4.1) means that the quantity ξ, defined by (3.5), undergoes the increment equal to π when the particle comes from the perigee ($r_{\min}$) to the apogee ($r_{\max}$) point. With the choice $r_1 = r_{\min}$ in equation (3.5) condition (4.1) restricts the motion of the Runge–Lenz vector (3.4) in such a way that it is (anti)parallel to the position vector of the particle every time the latter passes its (apogee) perigee point (see Fig. 1). Thus following

the angular precession of the perihelion the Runge–Lenz vector rotates. Recall that its length is an integral of motion according to (2.2), (2.9), (3.8). For the Coulomb case ($V(r) = -\alpha/r$) the function F vanishes and requirement (4.1) is fulfilled if the integral is calculated.

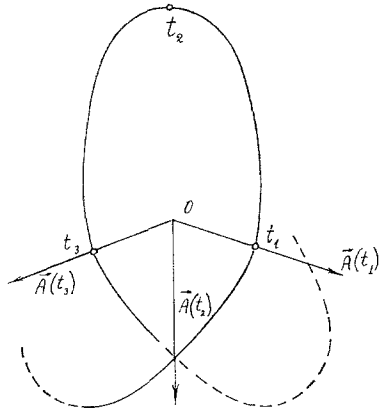

Figure 1.—Runge–Lenz vector meets the particle every time it passes the perigee points when travelling along the nonstationary orbit.

Expressing F from (4.1) one obtains

$$F(H,L^2) = -\frac{(\partial S/\partial L^2) + (\pi/2L)}{\partial S/\partial H} \tag{4.3}$$

where S is the action integral:

$$S(H,L^2) = \int_{r_{\min}}^{r_{\max}} (2(H - V(r))r^2 - L^2)^{1/2} \frac{dr}{r} \tag{4.4}$$

Now we may regard equation (3.6) as a first-order partial derivative differential equation for the Casimir invariant $G(H,L^2)$. With the use of equation (4.3) for F the general solution of equation (3.6) may be written in the form

$$S + \pi L = \Omega(G) \tag{4.5}$$

where Ω is arbitrary function of G. To specify this function one needs to formulate another requirement that would provide a boundary condition for equation (3.6).

In the (H,L^2)-plane the boundary of classical motion $H_B(L^2)$ is provided by the circular orbit. To get it one has to solve the equation

$$r_{\min} = r_{\max} \tag{4.6}$$

or, equivalently, to solve the equation

$$-\frac{dV}{dr}+\frac{L^2}{r^3}=0 \tag{4.7}$$

together with equation (4.2). Now we demand that the Runge–Lenz vector (3.4) vanish for the circular motion since in this limit there is no special direction in the plane of orbit. In the framework of nonbroken symmetry ($\{A_i, H\}=0$ instead of equation (2.3)) this property has been established by Bacry *et al.* (1966) and by the present authors (Serebrennikov & Shabad, 1971). It sufficed for unique determination of the Casimir invariant and was used by us in the paper cited to obtain l-degenerate energy spectrum for arbitrary central problem as an $O(4)$-symmetric approximation to the true quasiclassical spectrum.

Now imposing the requirement discussed implies:

$$G(H_B(L^2), L^2)=\frac{L^2}{2} \tag{4.8}$$

Taking into account the vanishing of expression (4.4) for the circular case one obtains from (4.5) the solution of equation (3.6) with the boundary condition (4.8):

$$S(H, L^2)+\pi L=\pi\sqrt{(2G)} \tag{4.9}$$

This is our final expression for the $O(4)$ Casimir invariant in terms of the Hamiltonian and angular momentum squared. If the action (4.4) can be calculated explicitly equation (4.9) allows determination of the function (2.10).

The quantisation prescription, according to Section 2 is to substitute $2G \to \hbar^2(n^2-1)$, $L^2 \to \hbar^2 l(l+1)$. If one makes besides the substitution $\hbar^2(n^2-1) \to \hbar^2 n^2$ and $\hbar^2 l(l+1) \to \hbar^2(l+\frac{1}{2})^2$, which is allowed within the accuracy of $\hbar$, one comes to the Bohr–Sommerfeld quantisation rule:

$$S(H, \hbar^2(l+\tfrac{1}{2})^2)=\pi\hbar(n-l-\tfrac{1}{2})=\pi\hbar(n_r+\tfrac{1}{2}) \tag{4.10}$$

where we put $n=n_r+l+1$ in agreement with (2.12).

Note that the only restrictions on the potential used throughout this section are imposed by the presence of the region of finite-range classical motion. The strongly singular potentials showing the fall down into the centre are also untreatable.

It is known that the Bohr–Sommerfeld quantisation gives the $\hbar \to 0$ limit of the Schrödinger equation energy spectrum if n and l are large. Our extra interpretation of this fact reduces to the remark that the Casimir invariant G is a classical quantity (as well as is the orbital momentum L^2) and should be kept fixed while the limiting transition $\hbar \to 0$ is being carried not.

In this section we have derived the Bohr–Sommerfeld quantisation rules by imposing the two conditions on the Runge–Lenz vector. We can also make

the inverse development, proceeding from (4.10). Taking (2.12) into account and expressing n and l as $n = \sqrt{(2G)}/h$, $l = L/h$, we obtain the classical Casimir invariant in the form (4.9) (h should be neglected everywhere). Now one can see that our requirements (4.8) and (4.1) are satisfied. We conclude, that these requirements imposed within the broken $O(4)$ scheme described in Section 2 are completely equivalent to the Bohr–Sommerfeld quantisation.

5. *Examples*

Here we consider several potentials and give explicit expressions for the corresponding Hamiltonians in terms of the Casimir invariant and the angular momentum squared. For the examples considered in this section we trace explicitly how the limiting transition $h \to 0$ in the exact quantum spectra gives the results coinciding with those of our method.

A. *Potential is* $V(r) = -\alpha/r + \beta/r^2$; $\alpha, \beta > 0$

One obtains from (4.9):

$$H = \frac{p^2}{2} - \frac{\alpha}{r} + \frac{\beta}{r^2} = -\frac{\alpha^2}{2} \frac{1}{(\sqrt{(2G)} - L + \sqrt{(L^2 + 2\beta)})^2} \tag{5.1}$$

The substitution $2G = h^2(n^2 - 1)$, $L^2 = h^2 l(l+1)$ gives the spectrum

$$E_{n,l} = -\frac{\alpha^2}{2} \frac{1}{(\sqrt{(h^2(n^2-1))} - \sqrt{(h^2 l(l+1))} + \sqrt{(h^2 l(l+1) + 2\beta)})^2} \tag{5.2}$$

The exact spectrum, obtained by solving the Schrödinger equation may be represented as

$$E_{n,l} = -\frac{\alpha^2}{2} \times \frac{1}{(\sqrt{(h^2(n^2-1) + h^2)} - \sqrt{(h^2 l(l+1) + h^2/4)} + \sqrt{(h^2 l(l+1) + h^2/4 + 2\beta)})^2} \tag{5.3}$$

where $n = n_r + l + 1$, $n_r = 0, 1, 2 \ldots$. It can be readily seen that (5.3) reduces to (5.2) if the limit $h \to 0$ is carried out with $2G = h^2(n^2 - 1)$ and $L^2 = h^2 l(l+1)$ kept fixed. Note that the traditional replacement of $l(l+1)$ by $(l + \frac{1}{2})^2$ together with the replacement of $(n^2 - 1)$ by n^2 turns the quasi-classical result (5.2) into the exact quantum spectrum (5.3). This fact, however, is out of the scope of the present consideration. In the limit $\beta = 0$ one obtains the Coulomb potential $V(r) = -\alpha/r$, possessing the symmetry ($\partial G/\partial L^2 = 0$). Equation (5.2) gives the degenerate spectrum

$$E_n = -\frac{\alpha^2}{2h^2(n^2-1)}$$

The spectrum could be given the exact quantum form $E_n = -(\alpha^2/2h^2 n^2)$ if

one took into account that in this case the $O(4)$ is an exact quantum symmetry group.

B. *Potential is* $V(r) = \omega^2 r^2/2 + \beta/r^2$, $\beta \geqslant 0$

According to (4.9) the Hamiltonian has the form

$$H = \frac{p^2}{2} + \frac{\omega^2 r^2}{2} + \frac{\beta}{r^2} = \omega(\sqrt{(L^2 + 2\beta)} - 2L + 2\sqrt{(2G)}) \tag{5.4}$$

and the spectrum is

$$E_{n,l} = \hbar\omega\left(\sqrt{\left(l(l+1) + \frac{2\beta}{\hbar^2}\right)} - 2\sqrt{l(l+1)} + 2\sqrt{(n^2-1)}\right) \tag{5.5}$$

which should be compared with the true quantum spectrum

$$\begin{aligned} E_{n,l} &= \hbar\omega\left(\sqrt{\left((l+\tfrac{1}{2})^2 + \frac{2\beta}{\hbar^2}\right)} + 2n_r + 1\right) \\ &= \hbar\omega\left(\sqrt{\left((l+\tfrac{1}{2})^2 + \frac{2\beta}{\hbar^2}\right)} - 2l - 1 + 2n\right) \end{aligned} \tag{5.6}$$

where $n = n_r + l + 1$, $n_r = 0, 1, 2\ldots$. Substituting here n and l expressed as

$$n = \sqrt{\left(\frac{2G}{\hbar^2} + 1\right)}, \qquad l = \frac{1}{2}\left(\sqrt{\left(1 + \frac{4L^2}{\hbar^2}\right)} - 1\right)$$

and putting $\hbar = 0$ we obtain (5.4). Note again that the change $l(l+1) \to (l+\frac{1}{2})^2$ and $n^2 - 1 \to n^2$ turns (5.5) into the exact expression (5.6).

Consider now isotropic harmonic oscillator $\beta = 0$. There are two perigee points on the closed elliptical orbit, since the centre of force coincides with the centre of the ellipse. The Runge–Lenz vector (3.4) meets the particle twice a period, while rotating with the same angular velocity as the position vector does but in the opposite direction. It takes a very simple form

$$\mathbf{A} = \frac{1}{2}\sqrt{\left(\frac{H + 3\omega L}{2\omega^2}\right)}\left(-\mathbf{r}\omega + \frac{\mathbf{p} \times \mathbf{L}}{L}\right) \tag{5.7}$$

The Poisson brackets between equation (5.7) and the expression $2\omega\sqrt{(2G)} = H + \omega L$ vanish. The function (3.6), (4.3) is $F = -(\omega/2L)$. The Hamiltonian (5.4) reduces to

$$H = \frac{p^2}{2} + \frac{\omega^2 r^2}{2} = \omega(2\sqrt{(2G)} - L) \tag{5.8}$$

and the spectrum (5.5) becomes

$$E_{n,l} = \hbar\omega(2\sqrt{(n^2-1)} - \sqrt{(l(l+1))}) \tag{5.9}$$

while the exact spectrum is

$$E_{n_r,l} = \hbar\omega(2n_r + l + \tfrac{3}{2}) \tag{5.10}$$

It is well known (Jauch & Hill, 1940) that the oscillatory problem possesses the exact $SU(3)$ symmetry. One $SU(3)$ multiplet joins the levels with the same value of the 'oscillatory principle quantum number' $n_{\text{osc}} = 2n_r + l + 1$, the energy (5.10) being degenerate within this multiplet: $E_{n_r, l} = \hbar\omega(n_{\text{osc}} + \frac{1}{2})$, $n_{\text{osc}} = 1, 2\ldots$. On the contrary, the $O(4)$ multiplet joins the levels with the 'hydrogen principle quantum number' $n = n_r + l + 1$ kept fixed and the energy is different for different members of the multiplet since $O(4)$ is not

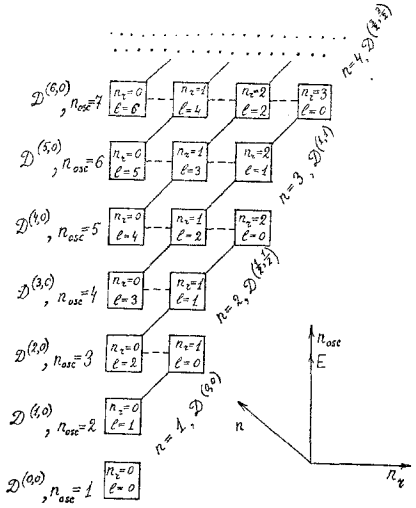

Figure 2.—Comparative classification of the oscillator levels via $SU(3)$ and $O(4)$ groups. Squares denote the $O(3)$ degenerate submultiplets. Each horizontal (dashed) line connects the members of one $SU(3)$ multiplet $\mathcal{D}(n_{\text{osc}} - 1, 0)$, $n_{\text{osc}} = 2n_r + l + 1$. Each sloped (solid) line connects the members of one $O(4)$ multiplet $\mathcal{D}[(n-1)/2, (n-1)/2]$, $n = n_r + l + 1$.

a symmetry group of this problem. Now (5.10) takes the form $E_{n_r, l} = \hbar\omega(2n - l - \frac{1}{2})$, $n = 1, 2\ldots$, comparable with (5.9) according to the rules above described. One $O(4)$ multiplet with a given n includes the levels belonging to several $SU(3)$ multiplets (see Fig. 2). Namely, the n_rth member of the nth $O(4)$ multiplet belongs also to the $SU(3)$ multiplet with $n_{\text{osc}} = n + n_r$. Vice versa, the level n_r, of the $SU(3)$ multiplet characterised by the definite value of n_{osc} belongs to $O(4)$ multiplet with $n = n_{\text{osc}} - n_r$.

C. *Finite Depth Well Potential*

As an example with a finite number of stable bound states consider *the finite depth square potential well*:

$$V(r) = \begin{cases} 0, & r < r_0 \\ V_0 > 0, & r > r_0 \end{cases}$$

Equation (3.6) should be considered in the region bounded by the circular

orbit condition $H_B = L^2/2r_0^2$ and by the condition $H < V_0 + L^2/2r_0^2$, which prevents the particle from escaping to infinity. Equation (4.9) gives

$$\pi\sqrt{(2G)} = \sqrt{(2Hr_0^2 - L^2)} + L\left[\pi - \arccos\left(\frac{L}{r_0\sqrt{(2H)}}\right)\right] \tag{5.11}$$

which also can be obtained as the $h \to 0$ limit of the exact quantum equation for the energy spectrum after substitution of $n_r = n - l - 1$, $n \to \sqrt{(2G)}/h$, $l \to L/h$. Consider the S-states: $L^2 = 0$. Then equation (5.11) gives the spectrum ($n = n_r + 1$):

$$E_{n,0} = h^2 \frac{\pi^2}{2r_0^2}(n^2 - 1) \tag{5.12}$$

where n may not exceed the value $\sqrt{((2V_0 r_0^2/h^2\pi^2) + 1)}$ since otherwise the energy would fall outside the above boundary $H = V_0 + L^2/2r_0^2$. Every s-state $l = 0$ is a beginner of an $O(4)$ multiplet of levels $0 \leqslant l \leqslant n - 1$, some of them being in fact quasistable. The levels of a given multiplet are inside the above region if the beginner is there thanks to the fact that the condition $L^2 \leqslant 2G$ or $l(l+1) \leqslant n^2 - 1$ admits all the values $0 \leqslant l \leqslant n - 1$ (remember that the equality $L^2 = 2G$ relates to the circular orbit).

6. *Inverse Problem*

In this section we discuss briefly the question of whether the potential can be restored if the Hamiltonian is originally given as a function (2.10) of the invariant operator of the Group (G) and of the symmetry breaking operator (L^2).

It is clear that the Hamiltonian cannot be found in the form (2.1), if an arbitrary function (2.10) is given, since the latter contains two independent parameters (G and L^2), whereas the potential $V(r)$ contains only one (r). It is possible, however, to take $H(G,L^2)$ as arbitrary function $K(L^2) = H(L^2/2,L^2)$ on the circular orbit where $2G = L^2$ in accord with (4.8). In this case potential can be restored almost uniquely. For doing this it is sufficient to solve the following differential equation for $V(r)$:

$$\frac{1}{2}\frac{dV}{dr}r + V(r) = K\left(r^3\frac{dV}{dr}\right) \tag{6.1}$$

For obtaining this we expressed L^2 from equation (4.7) and H from equation (4.2) (recall that the set of equations (4.2), (4.7) determines the circular orbit). The change of independent variable $u = -(2r^2)^{-1}$ reduces equation (6.1) to a standard form of the differential equation of Clairaut type which can be always solved in parametric form. After the potential is found the function $H(G,L^2)$ can be determined according to the rules of Section 4 also off the circular orbits. We therefore have a tool to verify whether a given broken symmetry construction may be produced by a potential and to find this potential (we mean the potential acting in some fictitions (e.g. isotopic) space).

7. *Concluding Remarks*

In the present paper we have developed a broken $O(4)$ symmetry approach to every central problem. This approach is dynamically realised within the quasiclassical quantisation. We believe, however, that exact quantum problems with arbitrary potential can also be dealt with via the same algebraic frame. The main practical obstacle for doing this is the lack of isomorphism between the Poisson brackets algebra and that of commutators. The thing to be done is to find an appropriate definition of the operators obtained by the substitution of the noncommuting quantities $\hat{\mathbf{r}}$ and $\hat{\mathbf{p}}$ for the canonical variables $\mathbf{r}$ and $\mathbf{p}$ in the classical expressions for the $O(4)$ generators, obtained in Sections 3 and 4. This should result in operators that on the one hand coincide with the above classical expressions in the limit $h = 0$ (the correspondance principle) and on the other hand preserve the $O(4)$ algebra and the conservation of its Casimir operator. This problem is far from being simple: up to now its solution is known only for the Coulomb problem where the appropriate definition is the Weyl symmetrisation of $\hat{\mathbf{r}}$ and $\hat{\mathbf{p}}$. A solution can also be given for the quantum oscillator (to be published elsewhere). In the general case a natural program is to try to formulate some step-by-step procedure. At every step we must have some approximate prescription for ordering operators inside the expressions of the Runge–Lenz vector to provide the necessary $O(4)$ commutation rules up to some power of h. This procedure is expected to be formulated within one or another version of the WKB method, and may be useful for this method by attributing an algebraic frame to it.

On the other hand one may expect some applications of the present results prior to making them more accurate. This hope is due to the fact that our (quasiclassical) approximation is physically interesting. One should remember that the broken symmetry or the current algebra approaches in the theory of elementary particles sometimes work within this accuracy (compare, e.g. the tree approximation in the chiral dynamics).

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Appendix

Here we present a proof for the Hamiltonian to be a function of three independent integrals of motion G, L^2, L_3 if the Poisson brackets among them vanish. When these quantities are given as functions of the coordinates r_1, r_2, r_3 and momenta p_1, p_2, p_3, one may express the p's as $p_i = f_i(G, L^2, L_3, r_1, r_2, r_3)$ provided there is not a single relation interconnecting the quantities G, L^2, L_3, r_1, r_2, r_3. We refer to this condition as an independence of G, L^2, L_3 and always assume it to be fulfilled. On

eliminating the momenta from the Hamiltonian (2.1) one may express the partial derivative $(\partial H/\partial r_i)_{G,L^2,L_3}$ calculated with G, L^2, L_3 kept fixed, as follows:

$$\left(\frac{\partial H}{\partial r_i}\right)_{G,L_2,L_3} = \frac{\partial H}{\partial r_i} + \frac{\partial H}{\partial p_k}\frac{\partial f_k}{\partial r_i}$$
$$= -\frac{dp_i}{dt} + \frac{dr_k}{dt}\frac{\partial f_k}{\partial r_i} = \frac{dr_k}{dt}\left(\frac{\partial f_k}{\partial r_i} - \frac{\partial f_i}{\partial r_k}\right) \tag{A.1}$$

In deriving (A.1) we used the Hamilton equations of motion and the fact that G, L^2, L_3 are the integrals of motion. It is known (see Whittaker (1944)) that the Poisson brackets $\{(p_i - f_i), (p_k - f_k)\}$, $i, k = 1, 2, 3$ vanish if the brackets of G, L^2, L_3 do so. Consequently, $(\partial f_i/\partial r_k) - (\partial f_k/\partial r_i) = 0$ and the partial derivative (A.1) vanishes. We conclude that the Hamiltonian does not depend upon r_i after substitution of $p_i = f_i(G, L^2, L_3, \mathbf{r})$. This completes the proof.

Note that G, L^2, L_3 may be referred to as new coordinates, say Q_1, Q_2, Q_3. Due to the above property $(\partial f_k/\partial r_i) = (\partial f_i/\partial r_k)$ the expression $\sum p_i dr_i$ is a perfect differential and thus a contact transformation from the set r_i, p_i to the set $Q_i, \mathscr{P}_i$ can be defined. It is sufficient to define a generating function $W(r_1, r_2, r_3, Q_1, Q_2, Q_3)$ as a solution of the equations $\partial W/\partial r_i = f_i(r_1, r_2, r_3, Q_1, Q_2, Q_3)$. The new momenta are then $\mathscr{P}_i = -(\partial W/\partial Q_i)$. The Hamiltonian remains unchanged under this contact transformation since $(\partial W/\partial t) = 0$, and its independence upon the $\mathscr{P}$'s is due to the invariance of the Poisson brackets under the contact transformation $\{H, Q_i\} = -\partial H/\partial \mathscr{P}_i = 0$.

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